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Effective equations for flow in random porous media with a large number of scales

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We rederive Brinkman's equations for the flow of slow viscous fluid past a random distribution of identical obstacles using the Foldy's approximation. It is shown explicitly that such a derivation is valid for extremely dilute systems. We argue that Brinkman's equation can be used even in systems with lower porosity by proposing a model of porous media that has a very large number of scales.

1. Introduction

One of the most important equations used to describe flow in porous media is Brinkman's equation. It was suggested by Brinkman (1947) and later it was realized by Tam (1969) that the proper way to derive that equation is through the Foldy's (1945) approximation. Brinkman's equation has become a major tool in the theoretical investigation of flow in porous media especially in the analysis of interface conditions (Saffman 1971; Haber & Mauri 1983) and long-range hydrodynamical interaction (Childress 1972; Saffman 1973; Kim & Russel 1985). In view of the many applications of this equation, we find it worthwhile to look more carefully at its derivation. Tam (1969) referred to the approximation he was using, saying '...we will assume that it is valid although we cannot prove it to be so'. A rigorous proof for the convergence of Stokes equation in domains containing random distributions of identical spheres to the Brinkman's equation was given recently (Rubinstein 1986). Such a convergence is achieved when the number of obstacles N tends to infinity as their radii tend to zero with a rate of $O(N^{-1})$. In fact, this special scaling appears implicitly in Tam's calculations, though is is not transparent through the formulation he used. According to the scaling we impose, it turns out that Brinkman's equation is the correct effective equation for very dilute systems, since ϕ (the volume fraction occupied by the obstacles) behaves like N^{-2}. On the other hand, drag predictions based on Brinkman's equation are done for systems which are not so extremely dilute.

The aim of this paper is twofold: first we show explicitly why the special scaling mentioned above is so crucial for the Foldy's approximation; and then we propose a model for porous media which has a large number of microscopic scales, and for which the modified Foldy's approximation yields Brinkman's equation when ϕ is small but still much larger than $O(N^{-2})$.

In §2 we derive the effetive equation using a formulation that was introduced by Papanicolaou (1985) for diffusion problems, and in §3 we analyse the many-scales model. Our methods are (mathematically speaking) formal. Rigorous proof are quite complicated and will be given elsewhere.

2. Foldy's approximation for identical spheres

We consider the flow of slow viscous fluid past a distribution of N identical spherical obstacles. The spheres are centred at $\{y_j\}$, $j = 1, 2, \ldots, N$, and have radius a ($a \ll 1$). They are assumed to remain fixed at $\{y_j\}$. The points $\{y_j\}$ are randomly distributed, independent of each other with a density probability function $\rho(x)$ that is taken to be continuous and it vanishes outside some finite domain. We scale this domain to have volume 1. The flow is described by the stokes equations

$$\left.\begin{aligned} \mu \Delta \boldsymbol{u}^N &= \nabla p^N + \boldsymbol{f} \quad \text{in } R^3 - B^N, \\ \nabla \cdot \boldsymbol{u}^N &= 0 \quad \text{in } R^3 - B^N, \\ \boldsymbol{u}^N &= 0 \quad \text{for } |\boldsymbol{x} - \boldsymbol{y}_j| = a \quad (j = 1, 2, \ldots, N). \end{aligned}\right\} \tag{2.1}$$

Here B^N is the domain occupied by the spheres, and $\boldsymbol{f}$ is a given body force.

The fundamental solution (Stokeslet) for the Stokes equation is

$$W_{ij} = \frac{1}{8\pi\mu}\left[\frac{\delta_{ij}}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{(x_i - y_i)(x_j - y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right] \tag{2.2}$$

(Happel & Brenner 1965) where the singularity is at $\boldsymbol{y}$.

We can write the solution to (2.1) in terms of W_{ij} as

$$u_i^N = \int_{R^3 - B^N} W_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \sum_{k=1}^{N} \int_{|\boldsymbol{y}-\boldsymbol{y}_k| = a} W_{ij}(\boldsymbol{x}, \boldsymbol{y}) \left(\frac{\partial u_j^N}{\partial n}(\boldsymbol{y}) - p(\boldsymbol{y})\, n_j\right) \mathrm{d}s_k. \tag{2.3}$$

The first approximation is the point-sources approximation: we replace $W_{ij}(\boldsymbol{x}, \boldsymbol{y})$ for $|\boldsymbol{y} - \boldsymbol{y}_k| = a$ by $W_{ij}(\boldsymbol{x}, \boldsymbol{y}_k)$, and write

$$u_i^N \approx \int_{R^3} W_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \sum_{k=1}^{N} W_{ij}(\boldsymbol{x}, \boldsymbol{y}_k) \int_{|\boldsymbol{y}-\boldsymbol{y}_k| = a} \left(\frac{\partial u_j^N}{\partial n}(\boldsymbol{y}) - p(\boldsymbol{y})\, n_j\right) \mathrm{d}s_k. \tag{2.4}$$

(Notice that we extended the domain of the first integral to R^3, assuming that $\phi = \frac{4}{3}\pi N a^3$ is small.)

The second step is to approximate the surface integrals in (2.4). We assume that as N becomes large, $\boldsymbol{u}^N$ tends to a smooth vector field $\boldsymbol{v}$ and, furthermore, each sphere 'feels' in its vicinity (i.e. on a scale which is comparable with its own lengthscale a) the following local problem:

$$\left.\begin{aligned} \mu \Delta \boldsymbol{u}^N &= \nabla p^N, \\ \nabla \cdot \boldsymbol{u}^N &= 0, \\ \boldsymbol{u}^N &= 0, \quad \text{for } |\boldsymbol{x} - \boldsymbol{y}_k| = a, \\ \boldsymbol{u}^N \to \boldsymbol{v}(\boldsymbol{x}) &\approx \boldsymbol{v}(\boldsymbol{y}_k) \quad \text{for } |\boldsymbol{x} - \boldsymbol{y}_k| \gg a. \end{aligned}\right\} \tag{2.5}$$

In the last step we have assumed that $\boldsymbol{v}(\boldsymbol{x})$ does not change much over distances a. Then, the surface integrals can be evaluated using Stokes formula and we obtain

$$u_i^N \approx \int_{R^3} W_{ij}(\boldsymbol{x}, \boldsymbol{y}) f_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + 6\pi\mu a \sum_{k=1}^{N} W_{ij}(\boldsymbol{x}, \boldsymbol{y}_k)\, v_j(\boldsymbol{y}_k). \tag{2.6}$$

Similarly, we can approximate p^N by

$$p^N \approx \int_{R^3} \pi_j(\boldsymbol{x}, \boldsymbol{y}) f_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + 6\pi\mu a \sum_{k=1}^{N} \pi_j(\boldsymbol{x}, \boldsymbol{y}_k)\, v_j(\boldsymbol{y}_k), \tag{2.7}$$

where
$$\pi_j = \frac{1}{4\pi}\frac{x_j - y_j}{|x-y|^3}.$$

From our assumptions on $\{y_j\}$ we know that

$$\frac{1}{N}\sum_{k=1}^{N} g(y_k) \to \int \rho(y)\, g(y)\, \mathrm{d}y \quad \text{as } N \to \infty \tag{2.8}$$

for every continuous function $g(y)$. (In fact, the last statement holds in probability 1 by the law of large numbers.) So if we want to get a non-trivial limit in (2.6) or (2.7) we must choose

$$a = \frac{\gamma}{N} \quad (\gamma = O(1)) \tag{2.9}$$

Taking now the limits (i.e. smoothing (2.6) and (2.7) by (2.8)) and identifying those limits as $v(x)$ and $p(x)$ (the effective velocity and pressure respectively), we get

$$v_i(x) = \int_{R^3} W_{ij}(x,y) f_j(y)\, \mathrm{d}y + 6\pi\mu\gamma \int_{R^3} W_{ij}(x,y)\, v_j(y)\, \mathrm{d}y, \tag{2.10}$$

$$p(x) = \int_{R^3} \pi_j(x,y) f_j(y)\, \mathrm{d}y + 6\pi\mu\gamma \int_{R^3} \pi_j(x,y)\, v_j(y)\, \rho(y)\, \mathrm{d}y, \tag{2.11}$$

which are the integral representation of Brinkman's equation

$$\left.\begin{aligned} \mu\,\Delta v - 6\pi\mu\gamma\rho(x)\, v &= \nabla p + f, \\ \nabla\cdot v &= 0, \end{aligned}\right\} \tag{2.12}$$

The formal approximation that we employed was proved to converge under very general conditions for diffusion problems by Papanicolaou & Varadhan (1980), Ozawa (1983) and Figari, Orlandi & Teta (1985) (among others), and for the Stokes equation by Rubinstein (1986). As we explained in the introduction, the scaling (2.9) implies

$$\phi = \tfrac{4}{3}\pi\gamma^3 N^{-2}, \tag{2.13}$$

i.e. we have a very dilute system. It turns out, however, that if one computes the drag on a test sphere, which is assumed to be immersed in a fluid governed by an effective field equation like (2.12), the results are in good agreement with experiments even for $\phi \approx 0.1$–0.2 (which is much higher than (2.13) for typical values of N). We believe that the reason for it is that in reality the medium does not consist just of identical obstacles, nor even of particles of the same size, but rather it is composed of obstacles whose dimensions are spread over many lengthscales. This is evidently true for packed configurations, but should also hold for more dilute systems. In the next section we propose a model for such media. This is by no means the only possible model, though its basic feature (i.e. there are many spheres on each scale, but their number decreases with increasing scale) should be common to all such models.

3. The multiple-scale model

We stretch now the domain of validity for Brinkman's equation to higher volume fractions. We propose the following model: Let $0 < \alpha < 1$, $\infty > \beta > 1$ be fixed constants independent of N_0 (the number of spheres.) The spherical obstacles are centred at $\{y_i\}$ as before. They come in groups where the ith group has N_i spheres with radius R_i, and

$$N_i = \alpha^i N, \quad R_i = \gamma\beta^i N^{-1} \quad (i = 0, 1, 2, \ldots, k(N)), \tag{3.1}$$

where N is a large number. (In order to make sense, we take the integer value of $\alpha^i N$.) We are interested in the limit $N \to \infty$ and we want $N_i(N) \to \infty$ as well. Hence we restrict $k(N)$ so that

$$N_k = \alpha^k N = N^\epsilon \quad \text{for some fixed } \epsilon > 0. \tag{3.2}$$

Finally, we relate α and β by

$$\beta = \alpha^{-1+\delta} \quad (\delta > 0). \tag{3.3}$$

Let us look at some consequences of the construction:

(i) The number of spheres N_0 satisfies

$$\frac{N_0}{N} = \sum_{i=0}^{k} \alpha^i \to \frac{1}{1-\alpha} \quad \text{as } N \to \infty,$$

i.e. $N_0 = O(N)$.

(ii) The volume fraction occupied by the obstacle is

$$\phi = \tfrac{4}{3}\pi N^{-2}\gamma^3 \sum_{i=0}^{k} \alpha^i \beta^{3i} = O(N^{-2\epsilon-3\delta}) \quad \text{for } N \text{ large.} \tag{3.4}$$

(iii) If we repeat Foldy's approximation, we find (cf. (2.6), (2.7))

$$u_i^N \approx \int W_{ij}(\boldsymbol{x},\boldsymbol{y}) f_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + 6\pi\mu\gamma \sum_{l=0}^{k(N)} \frac{\beta^l \alpha^l}{N_l} \sum_{n=1}^{N_l} W_{ij}(\boldsymbol{x},\boldsymbol{y}_n)\, v_j(\boldsymbol{y}_n), \tag{3.5}$$

$$p^N \approx \int \pi_j(\boldsymbol{x},\boldsymbol{y}) f_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + 6\pi\mu\gamma \sum_{l=0}^{k(N)} \frac{\beta^l \alpha^l}{N_l} \sum_{n=1}^{N_l} \pi_j(\boldsymbol{x},\boldsymbol{y}_n)\, v_j(\boldsymbol{y}_n). \tag{3.6}$$

Taking the limit $N \to \infty$, using (3.3) and the proposition discussed in the Appendix, we arrive at the effective equation

$$\mu\,\Delta \boldsymbol{v} - 6\pi\mu\gamma\, A(\alpha,\delta)\, \rho(\boldsymbol{x})\, \boldsymbol{v} = \nabla p + \boldsymbol{f}, \tag{3.7}$$

$$\nabla \cdot \boldsymbol{v} = 0,$$

$$A(\alpha,\delta) = \frac{1}{1-\alpha^\delta}. \tag{3.8}$$

4. Remarks

(i) By choosing δ and ϵ to be small (but positive) we can achieve volume fractions much higher than $O(N^{-2})$. We note, however, that taking ϵ small will result in a slow rate of convergence.

(ii) For $\delta \ll 1$, the second term on the left-hand side of (3.7) is dominant. This is of course the D'arcy limit, where the permeability K is given by

$$K = [6\pi\gamma\, A(\alpha,\delta)\, \rho(\boldsymbol{x})]^{-1}.$$

(iii) The crucial point in this construction was that $\sum (\alpha\beta)^i$ converges while $\sum \alpha^i \beta^{3i}$ diverges.

(iv) Our results should not be confused with higher-order correction terms which must be added to (2.12) when the volume fraction becomes so large that the point-sources approximation breaks down and higher multiples must be taken into account.

Appendix

Let $\{r_j\}$ be a collection of independent, identically distributed random variables with expectation μ and variance σ^2. Using the notation of §3, we want to estimate

$$\mathrm{Prob}\left\{\left|\sum_{l=0}^{k}\frac{\alpha^{\delta l}}{N_l}\sum_{j=1}^{N_l} r_j-\frac{1}{1-\alpha^{\delta}}\mu\right|>b\right\},$$

where $b>0$ is arbitrary. From Chebychev's inequality, this is controlled by

$$\frac{1}{b}\sum_{l=0}^{k}\frac{\alpha^{2\delta l}}{N_l^2}\sum_{j=0}^{N_l}\sigma^2\leqslant\frac{1}{b}\frac{\sigma^2}{N^{\epsilon}}\frac{1}{1-\alpha^{2\delta}}\to 0$$

as $N\to\infty$ for any fixed $b>0$, and so we can use

$$\lim_{N\to\infty}\sum_{l=0}^{K}\frac{\alpha^{\delta l}}{N_l}\sum_{j=1}^{N_l} r_j=\mu\frac{1}{1-\alpha^{\delta}}.$$

(Actually, this limit holds with probability 1.)

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